Causal symmetries

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Abstract. Based on the recent work [4] we put forward a new type of transformation for Lorentzian manifolds characterized by mapping every causal future-directed vector onto a causal future-directed vector. The set of all such transformations, which we call causal symmetries, has the structure of a submonoid which contains as its maximal subgroup the set of conformal transformations. We find the necessary and sufficient conditions for a vector field $\vec{\xi}$ to be the infinitesimal generator of a one-parameter submonoid of pure causal symmetries. We speculate about possible applications to gravitation theory by means of some relevant examples.

Our goal is to introduce a new type of spacetime symmetry which generalizes the conformal one while still preserving many causal properties of the Lorentzian manifolds. To that end, we will need the results on null-cone preserving maps analyzed and classified in [1]. The whole idea will be based on the new concept of *causal mapping* (leading to a definition of *isocausal spacetimes*) which was recently introduced in [4]. This letter is inspired by [1] and [4] which will be referred to as PI and PII from now on, respectively, and we use their notations. Herein, we will just give the fundamental results. A longer detailed exposition will be given elsewhere [5]. Some related ideas were used in [7].

According to PII, a causal relation between two Lorentzian manifolds is any diffeomorphism which maps non-spacelike (also called causal) future-directed vectors onto causal future-directed vectors. Here we will say that a transformation $\varphi : (V, \mathbf{g}) \to (V, \mathbf{g})$ is a *causal symmetry* if it sets a causal relation of $(V, \mathbf{g})$ with itself. From theorem 3.1 in PII follows that φ is a causal symmetry iff $\varphi^*\mathbf{g}$ satisfies the dominant energy condition, or in the notation of PI and PII, iff $\varphi^*\mathbf{g}$ is a future tensor: $\varphi^*\mathbf{g} \in \mathcal{DP}^+_2(V)$.

The set of causal symmetries of $(V, \mathbf{g})$ will be denoted by $\mathcal{C}(V, \mathbf{g})$ (in short $\mathcal{C}(V)$ if no confusion arises). This is a subset of the transformation group of V and clearly (prop. 3.3 of PII) the composition of causal symmetries is a causal symmetry. As the identity map is also a causal symmetry, $\mathcal{C}(V)$ has the algebraic structure of a submonoid, see e.g. [9]. Nonetheless $C(V)$ will not in general be a subgroup because the inverse of a causal relation need not be a causal relation. Actually, both φ and φ^{-1} are causal iff φ is a conformal transformation (theorem 4.2 of PII), and therefore the maximal subgroup $\mathcal{C}(V) \cap \mathcal{C}(V)^{-1}$ of $\mathcal{C}(V)$ [9] is just the group of conformal transformations of V: every subgroup of $\mathcal{C}(V)$ is formed exclusively by conformal symmetries. We call *proper causal symmetries* the causal symmetries which are not conformal transformations.

The set $\mathcal{C}(V)$ is invariant against conformal rescaling, that is $\mathcal{C}(V, e^{\sigma}\mathbf{g}) = \mathcal{C}(V, \mathbf{g})$ for all differentiable functions σ, so the assertion that φ is a causal symmetry is a conformally invariant one. Moreover, if $(V, \mathbf{g}_1)$ and $(V, \mathbf{g}_2)$ are isocausal —meaning that there are mutual causal relations ϕ and ψ, see PII—, then there is a one-to-one mapping between $\mathcal{C}(V, \mathbf{g}_1)$ and $\mathcal{C}(V, \mathbf{g}_2)$ because if $\varphi \in \mathcal{C}(V, \mathbf{g}_1)$ then one can easily construct a causal symmetry of $(V, \mathbf{g}_2)$ (say $\phi \circ \varphi \circ \psi$), and vice versa. These two facts allow us to claim that causal symmetries keep the causal structure —in the sense of PII— invariant.

For any non-zero rank-r future tensor $\mathbf{T} \in \mathcal{DP}^+_r(V)$ we define the set of its *principal null directions*, denoted by $\mu(\mathbf{T})$, as the set of future-directed vectors $\vec{k}$ such that $\mathbf{T}(\vec{k}, \dots, \vec{k}) = 0$. This immediately implies that $\vec{k}$, being causal, must in fact be null, (property 2.3 in PI). This concept is a close relative of the one presented in [11], which itself is a generalization of the principal null directions of the Weyl tensor. By definition, the set of canonical null directions (section 4 of PII) of a causal symmetry φ is simply $\mu(\varphi^*\mathbf{g})$, whose elements are the null eigendirections of $\varphi^*\mathbf{g}$. Then we have

$$\varphi \in \mathcal{C}(V) \Longrightarrow \varphi'[\mu(\varphi^*\mathbf{T})] \subseteq \mu(\mathbf{T}) \text{ and } \mu(\varphi^*\mathbf{T}) \subseteq \mu(\varphi^*\mathbf{g}),\ \forall \mathbf{T} \in \mathcal{DP}^+_r(V). \qquad (1)$$

Recall that if $\varphi \in \mathcal{C}(V)$, then $\varphi^*\mathbf{T} \in \mathcal{DP}^+_r(V)$ for all $\mathbf{T} \in \mathcal{DP}^+_r(V)$ (proposition 3.1 of PII). The first assertion follows immediately from $(\varphi^*\mathbf{T})(\vec{k}, \dots, \vec{k}) = \mathbf{T}(\varphi'\vec{k}, \dots, \varphi'\vec{k})$, and the second from the fact that $\varphi'\vec{k}$ is null if $\vec{k} \in \mu(\varphi^*\mathbf{T})$ —using again property 2.3 in PI—, so that $0 = \mathbf{g}(\varphi'\vec{k}, \varphi'\vec{k}) = (\varphi^*\mathbf{g})(\vec{k}, \vec{k})$. Important corollaries of (1) are (i) $\mu(\varphi^*\mathbf{g}) = \emptyset \implies \mu(\varphi^*\mathbf{T}) = \emptyset$ and (ii) $\mu(\mathbf{T}) = \emptyset \implies \mu(\varphi^*\mathbf{T}) = \emptyset$, $\forall \mathbf{T} \in \mathcal{DP}^+_r(V)$.

Of course, the μ-sets depend on the point of the manifold. However, using the techniques of algebraic decompositions of spacetimes [6, 13] one can see that V splits in open subsets where $\mu(\varphi^*\mathbf{g})$ has a constant number of linearly independent elements. Henceforth, we will work on one of these subsets and assume that $\mathbf{g}$ is analytic there.

As usual with general symmetries, we are interested in the possibility of constructing one-parameter groups of causal symmetries, and their infinitesimal versions. Let $\{\varphi_s\}_{s\in I}$ be a local one-parameter group of transformations where $I \subseteq \mathbb{R}$ is an open interval and s its canonical parameter. When do these groups contain elements of $\mathcal{C}(V)$? A first answer comes from the following fact: if $\{\varphi_s\}_{s\in[0,\epsilon)} \subset \mathcal{C}(V)$ with $[0,\epsilon) \subset I$, then every element of $\{\varphi_s\}_{s\in\mathbb{R}^+\cap I}$ is a causal symmetry. This follows because every $s_0 \in \mathbb{R}^+$ can be written as a finite sum of numbers $s_1, \dots, s_j \in [0, \epsilon)$, so that $\varphi_{s_0} = \varphi_{s_1+\dots+s_j} = \varphi_{s_1} \circ \dots \circ \varphi_{s_j}$ is a composition of causal symmetries and thus a causal symmetry itself.

Now suppose that under the above hypotheses φ_{s_0} is a conformal transformation for $|s_0| \in I \cap \mathbb{R}^+$ and let $\vec{k}$ be an arbitrary future-directed null vector. Since a conformal transformation maps null vectors onto null vectors, $\varphi'_{|s_0|}\vec{k}$ is also null. Then $\varphi'_s\vec{k}$ is null for all $s \in [0, |s_0|]$ because $\varphi'_{|s_0|}\vec{k} = \varphi'_{s_1}[\varphi'_{s_2}\vec{k}]$ with $|s_0| = s_1 + s_2$ where $s_1, s_2 \in (0, |s_0|)$, and using that $\varphi_{s_1}, \varphi_{s_2}$ are causal symmetries one has that $\varphi'_{s_2}\vec{k}$ is causal and then (proposition 3.2 of PII) it must necessarily be null, proving that φ_s are conformal transformations $\forall s \in (-|s_0|, |s_0|)$ since they map null vectors onto null vectors (theorem 4.2 of PII). In turn, this implies that $\{\varphi_s\}_{s\in I}$ consists of conformal symmetries due to the group property of such transformations. We summarize this in the next result.

Result 1 *Suppose $\{\varphi_s\}_{s\in I}$ is a local one-parameter group of transformations such that $\{\varphi_s\}_{s\in I\cap\mathbb{R}^+}\subset\mathcal{C}(V)$. Then either φ_s is a conformal transformation for every value of $s\in I$ or $\{\varphi_s\}_{s\in I}$ contains no conformal transformations other than the identity ($=\varphi_0$).*

An immediate corollary of this result is that there cannot be *cyclic* submonoids of proper causal symmetries, so that the orbits of these submonoids can never be closed. For if $\{\varphi_s\}_{s\in S^1}$ were an effective realization of the circle formed by causal symmetries, then with the usual parameterization $\varphi_{2\pi}$ would be the identity map, that is to say, a (conformal) isometry, so that the whole subgroup would be conformal. Obviously, we will be interested in cases *with* proper causal symmetries. The set $\{\varphi_s\}_{s\in I\cap\mathbb{R}^+}$ will be called a *local one-parameter submonoid of causal symmetries* if it is a subset of $\mathcal{C}(V)$.

Our first fundamental result regarding these submonoids is that for $s>0$ the sets $\mu(\varphi_s^*\mathbf{g})$ are independent of s, and their elements are simply the null vector fields which remain null under the action of $\{\varphi_s\}_{s\in I}$. To prove this, let $\vec{k}$ be a null future-directed vector in $\mu(\varphi_{s_0}^*\mathbf{g})$ for $s_0\in I\cap\mathbb{R}^+$, which is equivalent to $\varphi'_{s_0}\vec{k}$ being null and future-directed. Then, reasoning in much the same way as we did in Result 1, we get that $\varphi'_s\vec{k}$ is null future-directed $\forall s\in[0,s_0]$ which is only possible (proposition 4.1 of PII) if $\vec{k}\in\mu(\varphi_s^*\mathbf{g})$. Then, the analytic function $f_{\vec{k}}(s)\equiv(\varphi_s^*\mathbf{g})(\vec{k},\vec{k})=\mathbf{g}(\varphi'_s\vec{k},\varphi'_s\vec{k})$ vanishes on the open interval $(0,s_0)$ and hence it must vanish on the whole I. As a bonus we also deduce that $\varphi'_s\vec{k}$ is null for all $s\in I$. Let $\vec{\xi}$ be the infinitesimal generator of $\{\varphi_s\}_{s\in I}$. We denote simply by $\mu_{\vec{\xi}}$ the set $\mu(\varphi_s^*\mathbf{g})$ for *any* $s>0$ and their elements are called the *canonical null directions* of the submonoid of causal symmetries. All the elements of $\mu_{\vec{\xi}}$ are eigenvectors of $\varphi_s^*\mathbf{g}$ with the *same* eigenvalue λ_s for each s. From the above $\varphi'_s(\mu_{\vec{\xi}})=\mu_{\vec{\xi}}$ for every $s\in I$, which allows to get the following fundamental property.

Result 2 *If $\{\varphi_s\}_{s\in I\cap\mathbb{R}^+}\subset\mathcal{C}(V)$, then for every $\mathbf{T}\in\mathcal{DP}_r^+(V)$, $\varphi'_s[\mu(\varphi_s^*\mathbf{T})]=\mu(\mathbf{T})\cap\mu_{\vec{\xi}}$.*

The inclusion $\varphi'_s(\mu(\varphi_s^*\mathbf{T}))\subseteq\mu_{\vec{\xi}}\cap\mu(\mathbf{T})$ follows directly from (1) if we take into account that $\varphi'_s(\mu_{\vec{\xi}})=\mu_{\vec{\xi}}$, $\forall s\in I$. Conversely, pick up any $\vec{k}\in\mu_{\vec{\xi}}\cap\mu(\mathbf{T})$ so that $0=\mathbf{T}(\vec{k},\ldots,\vec{k})=(\varphi_s^*\mathbf{T})(\varphi'_{-s}\vec{k},\ldots,\varphi'_{-s}\vec{k})$ for $s>0$. As $\vec{k}\in\mu_{\vec{\xi}}$, $\varphi'_{-s}\vec{k}$ must be null for every $s\in I$, and since $\varphi_s^*\mathbf{T}\in\mathcal{DP}_r^+(V)$ we get that $\varphi'_{-s}\vec{k}\in\mu(\varphi_s^*\mathbf{T})$ from what $\mu_{\vec{\xi}}\cap\mu(\mathbf{T})\subseteq\varphi'_s(\mu(\varphi_s^*\mathbf{T}))$ follows. Result 2 implies that if $\mu_{\vec{\xi}}\cap\mu(\mathbf{T})=\emptyset$ then $\varphi_s^*\mathbf{T}$ has no principal null directions for every $s>0$, while if $\mu_{\vec{\xi}}\subseteq\mu(\mathbf{T})$ then $\mu(\varphi_s^*\mathbf{T})=\mu_{\vec{\xi}}$.

As $\mu_{\vec{\xi}}$ is a set of null directions, it is not a vector space. Nevertheless, we can pick up a maximum number of *linearly independent* null vector fields $\{\vec{k}_1,\ldots,\vec{k}_m\}$ belonging to $\mu_{\vec{\xi}}$, so that $Span\{\mu_{\vec{\xi}}\}$ is invariant under the linear transformations φ'_s, being the eigenspace associated to λ_s for $s>0$. The number $m\equiv$dim$(Span\{\mu_{\vec{\xi}}\})$ is intrinsic to the submonoid of causal symmetries. Let $\mathbf{\Omega}=\mathbf{k}_1\wedge\ldots\wedge\mathbf{k}_m$ be a characteristic m-form over $Span\{\mu_{\vec{\xi}}\}$, where $\mathbf{k}_1,\ldots,\mathbf{k}_m$ are the one-forms associated to $\vec{k}_1,\ldots,\vec{k}_m$. From the previous results it is easy to see that‡

$$\varphi_s^*\mathbf{\Omega}=\sigma_s\mathbf{\Omega},\quad\text{for some }\sigma_s\in C^\infty(V),\ \forall s\in I\qquad\Longleftrightarrow\qquad \pounds_{\vec{\xi}}\mathbf{\Omega}=\gamma\mathbf{\Omega}.\quad(2)$$

‡ In the cases $m=1,2$ we can further establish the property $\varphi'_s\vec{k}\propto\vec{k}$, $\forall\vec{k}\in\mu_{\vec{\xi}}$, as is obvious.

The set $\mu_{\vec{\xi}}$ plays a key role in the study of the causal symmetries. Furthermore, it allows to set up a convenient classification of causal (and more general) symmetries, according to the number m defined above. When $m = n$ we recover the conformal symmetries, while for $1 \leq m < n$, we can speak of $\frac{m}{n}$*-partly conformal* symmetries, as they leave invariant the m independent null directions within $Span\{\mu_{\vec{\xi}}\}$. This view will be further supported later by the equations of the infinitesimal causal symmetries. Thus, we have a classification of causal symmetries, which split up into $n+1$ different types according to whether $m = 0, \ldots, n$. This is a more justified and better defined algebraic classification than the one recently outlined in [8]. It also includes, for $m = 1$, the newly studied case of Kerr-Schild vector fields [3]. It is worth noting that the symmetries closer to conformal ones are those with $m = n-1$, rendering those with $m = 1$ —in particular those of [3]– as the "less conformal" among the partly conformal symmetries. We will also see that the case with $m = 1$ is degenerate within this classification.

We have to know how to compute $\mu_{\vec{\xi}}$ or the generalization of the conformal property $\pounds_{\vec{\xi}}\mathbf{g} \propto \mathbf{g}$ to the causal symmetries. To that end, we need a lemma.

Lemma 1 *Let $\{\mathbf{T}_s\}$ be a one-parameter family, differentiable in s, of rank-r (covariant) tensors such that $\mathbf{T}_{s_0} = 0$ for some fixed s_0. Assume that $\mathbf{T}_s \in \mathcal{DP}^+_r(V)$ for all $s \in [s_0, s_0+\epsilon)$. Then $d\mathbf{T}_s/ds|_{s=s_0} \equiv \dot{\mathbf{T}}_{s_0} \in \mathcal{DP}^+_r(V)$ (or its contravariant counterpart).*

To prove it, define functions $f_{\vec{u}_1,\ldots,\vec{u}_r}(s) \equiv \mathbf{T}_s(\vec{u}_1, \ldots, \vec{u}_r)$ where $\vec{u}_1, \ldots, \vec{u}_r$ are any future-directed causal vectors. Clearly $f_{\vec{u}_1,\ldots,\vec{u}_r}(s_0) = 0$ while $f_{\vec{u}_1,\ldots,\vec{u}_r}(s) \geq 0$ for $s \in [s_0, s_0+\epsilon)$, which immediately implies $0 \leq df_{\vec{u}_1,\ldots,\vec{u}_r}/ds|_{s_0} = \dot{\mathbf{T}}_{s_0}(\vec{u}_1, \ldots, \vec{u}_r)$. As a first application, we are now ready to get the sought expression of $\pounds_{\vec{\xi}}\mathbf{g}$.

Result 3 *There exists a smooth function α such that $(\pounds_{\vec{\xi}}\mathbf{g} - \alpha\mathbf{g}) \in \mathcal{DP}^+_2(V)$.*

Indeed, $\varphi^*_s\mathbf{g} \in \mathcal{DP}^+_2(V)$ for every $s \in \mathbb{R}^+ \cap I$, hence we can apply the canonical *decomposition theorem* (theorem 4.1 of PI) to such causal tensors to get

$$\varphi^*_s\mathbf{g} = \sum_{p=m}^{n} \mathbf{T}\{\mathbf{\Omega}_{[p]}(s)\} = \sum_{p=m}^{n-1} \mathbf{T}\{\mathbf{\Omega}_{[p]}(s)\} + A^2_s\,\mathbf{g}, \tag{3}$$

where A_s is a differentiable function such that $A_0 = 1$, and $\mathbf{T}\{\mathbf{\Omega}_{[p]}(s)\}$ are the superenergy tensors [12, 1] of adequate simple p-forms $\mathbf{\Omega}_{[p]}(s)$. The general formula for the superenergy tensor of a p-form $\mathbf{\Sigma}$ is [12, 1]:

$$T\{\mathbf{\Sigma}\}_{ab} = \frac{(-1)^{p-1}}{(p-1)!}\left(\Sigma_{ac_2\ldots c_p}\Sigma_b{}^{c_2\ldots c_p} - \frac{1}{2p}g_{ab}\Sigma_{c_1\ldots c_p}\Sigma^{c_1\ldots c_p}\right). \tag{4}$$

Each term appearing in equation (3) is in $\mathcal{DP}^+_2(V)$ and we have distinguished the extreme value $p = n$ because the corresponding tensor is proportional to the metric (PI). Therefore, the family $\mathbf{T}_s = \varphi^*_s\mathbf{g} - A^2_s\mathbf{g}$ satisfies the conditions of Lemma 1 with $s_0 = 0$ from what Result 3 follows with $\alpha \equiv dA^2_s/ds|_{s=0}$ by using that $\pounds_{\vec{\xi}}\mathbf{g} = d(\varphi^*_s\mathbf{g})/ds|_{s=0}$.

We can apply now the decomposition theorem 4.1 of PI to the future tensor $\pounds_{\vec{\xi}}\mathbf{g} - \alpha\mathbf{g}$. To do it, we must know the set $\mu(\pounds_{\vec{\xi}}\mathbf{g} - \alpha\mathbf{g})$. As $\mu(\varphi^*_s\mathbf{g}) = \mu_{\vec{\xi}}$, $\varphi^*_s\mathbf{g}$

always have the null vector fields of $\mu_{\vec{\xi}}$ as eigendirections so that we can consistently choose in (3) $\mathbf{\Omega}_{[m]}(s) \propto \mathbf{\Omega}$ for all $s \in I \cap \mathbb{R}^+$ if $m > 0$. Thus, we will use the notation $\mathbf{S} \equiv \mathbf{T}\{\mathbf{\Omega}\}$ from now on. From the results in PI, $\mathbf{S}^2$ is proportional to $\mathbf{g}$ so that we will also assume that $\mathbf{S}$ has been normalized if $m > 1$, that is, $S_{ac}S^c{}_b = \mathrm{g}_{ab}$. The case $m = 1$ is degenerate in the sense that $S_{ac}S^c{}_b = 0$, equivalent to $\mathbf{S} = \mathbf{k} \otimes \mathbf{k}$ where $\mathbf{k}$ is a representative of the unique canonical null direction. It is quite simple to deduce that the elements of $\mu_{\vec{\xi}}$ are among the null eigenvectors of $\pounds_{\vec{\xi}}\mathbf{g}$ by using that, for any $\vec{k} \in \mu_{\vec{\xi}}$, $\varphi_s^*\mathbf{g}(\cdot, \vec{k}) = \lambda_s \mathbf{g}(\cdot, \vec{k})$, $\forall s \in I$. This implies that $\mu_{\vec{\xi}} \subseteq \mu(\pounds_{\vec{\xi}}\mathbf{g} - \alpha\mathbf{g})$. Now, assume that there were a $\vec{k} \in \mu(\pounds_{\vec{\xi}}\mathbf{g} - \alpha\mathbf{g}) \setminus \mu_{\vec{\xi}}$. Then $\varphi'_s\vec{k}$ would be timelike for $s > 0$ so that, using e.g. Lemma 2.5 in PI, we could write $\varphi'_s\vec{k} = c_s\vec{k} + \vec{n}_s$ where the $\vec{n}_s$ are null and future directed and $c_s > 0$ such that $\vec{n}_0 = \vec{0}$, $c_0 = 1$. But then the family $\varphi'_s\vec{k} - c_s\vec{k}$ would satisfy the hypotheses of Lemma 1 with $s_0 = 0$, proving that there would be a function c such that $-\pounds_{\vec{\xi}}\vec{k} + c\vec{k}$ is future pointing. On the other hand, using that $\vec{k} \in \mu(\pounds_{\vec{\xi}}\mathbf{g} - \alpha\mathbf{g})$ we get $0 = \pounds_{\vec{\xi}}[\mathbf{g}(\vec{k}, \vec{k})] = 2\mathbf{g}(\pounds_{\vec{\xi}}\vec{k}, \vec{k})$ so that $-\pounds_{\vec{\xi}}\vec{k} + c\vec{k}$ and $\vec{k}$, being both future pointing and orthogonal to each other, would necessarily be null and proportional, leading to $\pounds_{\vec{\xi}}\vec{k} \propto \vec{k} \Longleftrightarrow \varphi'_s\vec{k} \propto \vec{k}$, which would mean $\vec{k} \in \mu_{\vec{\xi}}$ in contradiction. Thus, $\mu_{\vec{\xi}} = \mu(\pounds_{\vec{\xi}}\mathbf{g} - \alpha\mathbf{g})$ and we have

$$\pounds_{\vec{\xi}}\mathbf{g} = \alpha\mathbf{g} + \beta\mathbf{S} + \mathbf{Q} \tag{5}$$

where $\mathbf{Q}$ is a symmetric rank-2 future tensor such that $\mu(\mathbf{Q}) \supset \mu_{\vec{\xi}}$ whence (see PI)

$$Q_a{}^b\Omega_{ba_2\dots a_m} = \lambda\Omega_{aa_2\dots a_m}, \quad Q_a{}^c(\mathrm{g}_{cb} + S_{cb}) = \lambda(\mathrm{g}_{ab} + S_{ab}) \Longrightarrow \quad Q_a{}^cS_{cb} = Q_b{}^cS_{ca}$$

and $\beta > 0$, $\lambda \geq 0$ are smooth functions. The first equation comes from $Q_a{}^bk_b = \lambda k_a$ $\forall\vec{k} \in \mu_{\vec{\xi}}$, while the second follows because $\mathbf{g} + \mathbf{S}$ is the projector onto $Span\{\mu_{\vec{\xi}}\}$.

Relations (2) and (5) are the fundamental equations of this letter. They are "stable" under repeated application of $\pounds_{\vec{\xi}}$, that is to say, the structure of their right hand sides remains the same. This is clear for (2). To prove it for (5) we need to know the Lie derivatives of tensors of the type of $\mathbf{S}$ or $\mathbf{Q}$. For $\mathbf{S}$ this can be easily done by using its explicit expression $\mathbf{S} = \mathbf{T}\{\mathbf{\Omega}\}$ (eq. (4)) which meets the normalization requirements if we put $\Omega_{c_1\dots c_m}\Omega^{c_1\dots c_m} = 2m!(-1)^{m-1}$ when $m > 1$. Then, by means of (2) and (5) we readily arrive at

$$\pounds_{\vec{\xi}}S_{ab} = \alpha S_{ab} + \beta\mathrm{g}_{ab} + Q_{ac}S^c{}_b, \qquad (m > 1). \tag{6}$$

As is clear from their derivation, eqs.(2), (5) and (6) are not independent and, actually, (2,5) are equivalent to (5,6) where, due to the chosen normalization, one necessarily has $2\gamma = m(\alpha + \beta + \lambda)$ for $m > 1$. In the degenerate case $m = 1$, α, β and γ can be kept arbitrary and the equation replacing (6) is just $\pounds_{\vec{\xi}}\mathbf{S} = 2\gamma\mathbf{S} \Longleftrightarrow \pounds_{\vec{\xi}}\mathbf{k} = \gamma\mathbf{k}$ $(m = 1)$.

With regard to tensors of type $\mathbf{Q}$, we need an intermediate result which asserts that for two given future tensors $\mathbf{T}_1$ and $\mathbf{T}_2$ with $\mu(\mathbf{T}_1) = \mu(\mathbf{T}_2)$ and $\dim(Span\{\mu(\mathbf{T}_1)\}) \geq 1$ we can always find a positive α_{12} and a future tensor $\mathbf{R}_1$ such that $\mathbf{T}_1 = \alpha_{12}\mathbf{T}_2 + \mathbf{R}_1$. The proof is rather straightforward by noticing the existence of an orthonormal basis which diagonalizes both $\mathbf{T}_1$ and $\mathbf{T}_2$. Thus, since $\mu(\varphi_s^*\mathbf{Q}) = \mu_{\vec{\xi}}$ (use Result 2), we can apply this to the causal tensors $\varphi_{s_1}^*\mathbf{Q}$ and $\varphi_{s_2}^*\mathbf{Q}$ for $s_1, s_2 \in [0,\epsilon)$ and write

$$\varphi_{s_2}^*\mathbf{Q} = \alpha_{s_2,s_1}\varphi_{s_1}^*\mathbf{Q} + \mathbf{R}_{s_2,s_1}, \quad \mathbf{R}_{s_2,s_1} \in \mathcal{DP}_2^+(V)$$

where we can choose $\alpha_{s_1,s_1} = 1$ and $\mathbf{R}_{s_1,s_1} = 0$. Applying Lemma 1 to the family $\varphi_s^*\mathbf{Q} - \alpha_{s,s_1}\varphi_{s_1}^*\mathbf{Q}$ with s_1 fixed we get

$$\left(\left.\frac{d(\varphi_s^*\mathbf{Q})}{ds}\right|_{s=s_1} - \left.\frac{d\alpha_{s,s_1}}{ds}\right|_{s=s_1} (\varphi_{s_1}^*\mathbf{Q}) \right) \in \mathcal{DP}_2^+(V), \quad s_1 \in [0,\epsilon)$$

from what, by putting $s_1 = 0$, the desired result follows:

Result 4 *For every* $\mathbf{Q} \in \mathcal{DP}_2^+(V)$ *with* $\mu(\mathbf{Q}) \supseteq \mu_{\vec{\xi}} \neq \emptyset$ *there is a smooth function* ψ *such that* $\pounds_{\vec{\xi}}\mathbf{Q} - \psi\mathbf{Q} \in \mathcal{DP}_2^+(V)$ *and* $\mu(\pounds_{\vec{\xi}}\mathbf{Q} - \psi\mathbf{Q}) \supseteq \mu_{\vec{\xi}}$.

In particular, we can apply this result to $\pounds_{\vec{\xi}}\mathbf{g} - \alpha\mathbf{g}$ to get as a corollary the existence of functions $\alpha_1, \ldots, \alpha_r, \ldots$ for all natural $r \in \mathbb{N}$ such that

$$(\pounds_{\vec{\xi}} - \alpha_r)\cdots(\pounds_{\vec{\xi}} - \alpha_1)(\pounds_{\vec{\xi}} - \alpha)\mathbf{g} \in \mathcal{DP}_2^+(V), \;\; \forall r \in \mathbb{N}$$

where at any level r the set of principal null directions always includes $\mu_{\vec{\xi}}$. This is the required property on the stability of (5).

Two remarkable equations deducible from (5) and (6) are ($m > 1$)

$$\pounds_{\vec{\xi}} S^a{}_b = 0, \qquad \pounds_{\vec{\xi}}(\mathbf{g} + \mathbf{S}) = (\alpha + \beta + \lambda)(\mathbf{g} + \mathbf{S}).$$

These formulae support the claim that causal symmetries define partly conformal Killing vectors, being conformal on $Span\{\mu_{\vec{\xi}}\}$. As, on the other hand, $\pounds_{\vec{\xi}}(\mathrm{g}_{ab} - S_{ab}) = (\alpha-\beta)(\mathrm{g}_{ab} - S_{ab}) + Q_a{}^c(\mathrm{g}_{cb} - S_{cb})$ they will also be conformal on the orthogonal subspace $\perp Span(\mu_{\vec{\xi}})$ if and only if $Q_a{}^c(\mathrm{g}_{cb} - S_{cb}) \propto (\mathrm{g}_{ab} - S_{ab})$, which is only possible if $\mathbf{Q} \propto \mathbf{g}$. This is equivalent, by redefining α, to $Q_{ab} = 0$ (and hence $\lambda = 0$). Therefore we say that a causal symmetry is *pure* if $\mathbf{Q} = 0$. The general case with $\mathbf{Q}$ non-vanishing will be dealt with in [5]. Observe that the cases $m = n-1, n$ are always pure. The generating vector fields of pure causal symmetries satisfy then ($m \neq 1$)

$$\pounds_{\vec{\xi}}\mathrm{g}_{ab} = \alpha\mathrm{g}_{ab} + \beta S_{ab}, \qquad \pounds_{\vec{\xi}}S_{ab} = \alpha S_{ab} + \beta\mathrm{g}_{ab}. \tag{7}$$

In the degenerate situation $m = 1$, the pure case can also be defined by the vanishing of $\mathbf{Q}$ and the corresponding equations are

$$\pounds_{\vec{\xi}}\mathrm{g}_{ab} = \alpha\mathrm{g}_{ab} + \beta k_a k_b, \qquad \pounds_{\vec{\xi}}k_a = \gamma k_a \tag{8}$$

which include ($\alpha = 0$) the Kerr-Schild vector fields of [3]. Of course, α and β (and γ if $m = 1$) actually depend on $\vec{\xi}$, so they will be called the gauge functions as in [3]. In fact eqs.(7) (or (8)) are also sufficient, even if $\mathbf{S}$ is just a future tensor or if $\mathbf{k}$ is just causal:

Result 5 *A vector field* $\vec{\xi}$ *which satisfies (7) (respectively (8)) with* $\beta\mathbf{S} \in \mathcal{DP}_2^+(V)$ *and* $\dim(Span\{\mu(\mathbf{S})\} \neq 1)$ *(resp.* $\mathbf{S} = \mathbf{k}\otimes\mathbf{k}$ *with causal* $\mathbf{k}$*) generates a one-parameter submonoid of causal symmetries* $\{\varphi_s\}_{s\in I\cap\mathbb{R}^+}$ *with* $\mu_{\vec{\xi}} = \mu(\beta\mathbf{S})$.

To prove this when $m \neq 1$, we use the general formula $\varphi_s^*(\pounds_{\vec{\xi}}\mathbf{T}) = d(\varphi_s^*\mathbf{T})/ds$ which by integration immediately leads to $\varphi_s^*(\mathbf{g}+\mathbf{S}) = \exp\{\int_0^s \varphi_t^*(\alpha+\beta)dt\}(\mathbf{g}+\mathbf{S})$ and $\varphi_s^*(\mathbf{g}-\mathbf{S}) = \exp\{\int_0^s \varphi_t^*(\alpha-\beta)dt\}(\mathbf{g}-\mathbf{S})$, from where we deduce

$$\varphi_s^*\mathbf{g} = \exp\left\{\int_0^s \alpha(\varphi_t)dt\right\}\left[\cosh\left(\int_0^s \beta(\varphi_t)dt\right)\mathbf{g} + \sinh\left(\int_0^s \beta(\varphi_t)dt\right)\mathbf{S}\right]$$

which are clearly future tensors for all $s > 0$ if $\beta\mathbf{S} \in \mathcal{DP}_2^+(V)$, so that $\{\varphi_s\}_{s\in\mathbb{R}^+\cap I}$ is a submonoid of causal symmetries. The proof for the other case ($m = 1$) is analogous.

Observe that β must have a definite sign, implying that the vector fields satisfying (7) do not form a vector space, but only a *wedge* or *cone*, see [9], of a vector space. Nevertheless, the study of (7) and (8) has an interest on its own right, independently of the gauges signs, as they always define pure partly conformal symmetries (albeit possibly not causal) with a vector space structure. Its general study will be addressed elsewhere [5], but in the rest of the letter we give some preliminary results. First of all, (7) defines a Lie-algebra structure: if $\vec{\xi}_1$ and $\vec{\xi}_2$ comply with eqs.(7) with gauges $\alpha_{\vec{\xi}_1}$, $\beta_{\vec{\xi}_1}$ and $\alpha_{\vec{\xi}_2}$, $\beta_{\vec{\xi}_2}$ respectively then their Lie bracket $[\vec{\xi}_2, \vec{\xi}_1]$ also satisfies (7) with gauges

$$\alpha_{[\vec{\xi}_2,\vec{\xi}_1]} = \pounds_{\vec{\xi}_2}\alpha_{\vec{\xi}_1} - \pounds_{\vec{\xi}_1}\alpha_{\vec{\xi}_2}, \quad \beta_{[\vec{\xi}_2,\vec{\xi}_1]} = \pounds_{\vec{\xi}_2}\beta_{\vec{\xi}_1} - \pounds_{\vec{\xi}_1}\beta_{\vec{\xi}_2}.$$

A similar computation leads the same conclusion for the degenerate case $m = 1$*. These Lie algebras define the corresponding transformations groups whose generators satisfy (7) (or (8)) and they can be, in certain cases, infinite dimensional. These groups will contain submonoids of causal symmetries only when the gauges $\beta_{\vec{\xi}}$ have a sign. Thus, if $\vec{\xi}_1$ and $\vec{\xi}_2$ generate pure causal symmetries with $\mu_{\vec{\xi}_1} = \mu_{\vec{\xi}_2}$ then $\pm[\vec{\xi}_2, \vec{\xi}_1]$ will also be such a kind of generator only if $\beta_{[\vec{\xi}_2,\vec{\xi}_1]} = \pounds_{\vec{\xi}_2}\beta_{\vec{\xi}_1} - \pounds_{\vec{\xi}_1}\beta_{\vec{\xi}_2}$ does not vanish anywhere.

An example of physical relevance is provided by the so-called warped products, that is to say, Lorentzian manifolds of the form $V_1 \times \hat{V}$ with metrics of type $\mathbf{g} = \mathbf{g}^1 - R^2\hat{\mathbf{g}}$ where $\mathbf{g}^1, \hat{\mathbf{g}}$ are metrics on $V_1, \hat{V}$ respectively, and R is a non-vanishing function on V_1. Here we concentrate on the case where $(V_1, \mathbf{g}^1)$ is m-dimensional and Lorentzian so that

$$ds^2 = \mathrm{g}^1_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta - R^2(x^\gamma)d\mathbf{S}^2_{n-m}, \quad d\mathbf{S}^2_{n-m} = \hat{\mathrm{g}}_{AB}(x^C)dx^A dx^B$$

where $\{x^\gamma\}$ $(\alpha, \beta, \gamma = 0, \ldots, m-1)$ are coordinates on V_1 and $d\mathbf{S}^2_{n-m}$ is the positive-definite line-element of $(\hat{V}, \hat{\mathbf{g}})$ whose coordinates are $\{x^A\}$ $(A, B, C = m, \ldots, n-1)$. We seek the pure causal symmetries with $\mathbf{\Omega} = \rho\, dx^0 \wedge \ldots \wedge dx^{m-1}$ where $\rho = \sqrt{2\det(\mathbf{g}^1)}$ to meet the needed normalization. Equations (7) imply that $\vec{\xi}$ decomposes as $\vec{\xi} = \vec{\xi}_1 + \vec{\xi}_2$ with $\vec{\xi}_1 = \xi^\alpha(x^\gamma)\partial_\alpha$, $\vec{\xi}_2 = \xi^A(x^B)\partial_A$, and also

$$\pounds_{\vec{\xi}_2}\hat{\mathbf{g}} = \left(\alpha - \beta - R^{-2}\vec{\xi}_1(R^2)\right)\hat{\mathbf{g}}, \quad \pounds_{\vec{\xi}_1}\mathbf{g}^1 = (\alpha+\beta)\mathbf{g}^1 \implies \frac{1}{\rho}\pounds_{\vec{\xi}_1}\rho + \partial_\gamma\xi_1^\gamma = \frac{1}{2}m(\alpha+\beta).$$

Notice that $\vec{\xi}_1$ and $\vec{\xi}_2$ are conformal symmetries of $\mathbf{g}^1$ and $\hat{\mathbf{g}}$ respectively. The number of independent $\vec{\xi}$ depends on n, m and the particular $V_1, \hat{V}$, and it can be finite $(n-m, m > 2)$ or infinite (in some cases with $n - m \leq 2$ or $m = 1, 2$).

* Actually, this reasoning is independent of the properties of $\mathbf{S}$ (or $\mathbf{k}$), so that (7) (or (8)) define Lie algebras for *any* tensor field $\mathbf{S}$ (or *any* one-form $\mathbf{k}$).

Simple examples of the above are provided by n-dimensional Minkowski spacetime in Cartesian coordinates. Its pure causal symmetries with $\perp Span(\mu_{\vec{\xi}}) = \langle \partial_{x^{n-1}} \rangle$ ($m = n-1$) are given by $\vec{\xi} = \vec{\xi}_1 + F(x^{n-1})\partial_{x^{n-1}}$ where $\vec{\xi}_1$ is any conformal Killing vector of the $(n-1)$-dimensional Minkowski spacetime and F is arbitrary. Thus, in this case $\vec{\xi}$ depends on $n(n+1)/2$ parameters and a function of one coordinate. Finite-dimensional cases also appear, as of course the strictly conformal case with $m = n$. Another example arises by taking (say) $n = 6$ and $Span\{\mu_{\vec{\xi}}\} = \langle \partial_{x^0}, \partial_{x^1}, \partial_{x^2} \rangle$, so that $\vec{\xi}_1$ and $\vec{\xi}_2$ are conformal Killing vectors of the 3-spaces $Span\{\mu_{\vec{\xi}}\}$ and $\perp Span\{\mu_{\vec{\xi}}\}$, respectively. Hence, now the general $\vec{\xi}$ depends on $10 + 10 = 20$ arbitrary parameters.

A more interesting situation comprises the spherically symmetric spacetimes, in which $(\hat{V}, \hat{\mathbf{g}})$ has positive constant curvature and $(V_1, \mathbf{g}^1)$ is 2-dimensional so that $ds_1^2 = 2e^{f(u,v)}dudv$, $\mathbf{\Omega} = \sqrt{2}e^f du \wedge dv$ and $\mu_{\vec{\xi}} = \{\partial_u, \partial_v\}$. This has a clear physical interpretation for $\mu_{\vec{\xi}}$ are the *radial* null directions. The previous calculation particularizes now to $\vec{\xi}_1 = \xi^u(u)\partial_u + \xi^v(v)\partial_v$ with $\vec{\xi}_1(f) + \xi^u_{,u} + \xi^v_{,v} = \alpha + \beta$. Observe that the gauges are determined by the data $f(u,v), R(u,v)$ and the particular $\vec{\xi}_2$ and $\vec{\xi}_1$. Thus the general $\vec{\xi}$ depend on two arbitrary functions $\xi^u(u), \xi^v(v)$, plus the number of independent conformal Killing vectors $\vec{\xi}_2$ of the $(n-2)$-sphere. For instance, if $n = 4$ this conformal group has 6 independent parameters and is isomorphic to the Lorentz group.

Several other symmetries already appeared in the literature are also included in the causal symmetries, such as the conformal Killing vectors, the Kerr-Schild vector fields [3], some examples given in [7], or the transformations studied in [2, 10, 8].

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References

[1] Bergqvist G and Senovilla J M M 2001 *Class. Quantum Grav.* **18** 5299
[2] Bonanos S 1992 *Class. Quantum Grav.* **9** 697
[3] Coll B, Hildebrandt S R and Senovilla J M M 2001 *Gen. Rel. Grav* **33** 649
[4] García-Parrado A and Senovilla J M M 2003 *Class. Quantum Grav.* **20** 625
[5] García-Parrado A and Senovilla J M M, in preparation
[6] Hall G S 1996 *Class. Quantum Grav.* **13** 1479
[7] Harris S G and Low R J 2001 *Class. Quantum Grav.* **18** 27
[8] Hildebrandt S R 2002 *Gen. Rel. Grav.* **34** 65
[9] Hilgert J, Hofmann K H and Lawson J D 1989 *Lie groups, Convex Cones and Semigroups.* (Oxford Science Publications)
[10] Mars M 2001 *Class. Quantum Grav.* **18** 719
[11] Pozo J M and Parra J M 2002 *Class. Quantum Grav.* **19** 967
[12] Senovilla J M M 2000 *Class. Quantum Grav.* **17** 2799
[13] Senovilla J M M and Vera R 1999 *Class. Quantum Grav.* **16** 1185